The best path error for 10 Monte Carlo simulations of Alg. 2.

The best path error for EM, Alg. 2 and Alg. 3.

**Contribution**

This work presents a method to estimate the process noise variance for a non-linear dynamic system with high state dimension. The proposed method makes use of the expectation maximization algorithm, where the E-step is solved by linearisation.

**Introduction**

The performance of a non-linear filter hinges in the end on the accuracy of the assumed non-linear model of the process. In particular, the process noise covariance Q. For non-linear models, there is on-going research on using the expectation maximisation (EM) algorithm with a particle smoother to estimate the parameters. However, the particle smoother is not applicable for models with high state dimension. The idea here is to:

- Linearise the non-linear model.
- Use an extended Kalman smoother (EKS).

Let the model be given by

\[ x_{k+1} = F_k(x_k, u_k) + F'_k(x_k) v_k \]

where \( x_k \in \mathbb{R}^n \), \( y_k \in \mathbb{R}^m \), \( v_k \sim \mathcal{N}(0, Q) \) and \( e_k \sim \mathcal{N}(0, R) \). All model parameters are assumed to be known except for \( Q \in S_+^n \). Assume also that \( F_k(x_k) \) has the following structure

\[ F_k(x_k) = \begin{pmatrix} 0 \\ F'_k(x_k) \end{pmatrix} \]

This type of model structure is common for mechanical systems derived by Newton’s law or Lagrange’s equation.

**The E-step**

The expectation of the log-likelihood function \( \log p(y_k | y_{1:k}, x_{1:k}) \) is calculated using the EKS and it can be expressed as

\[
\Gamma(Q; Q_k) = \tilde{L} - \frac{1}{2} \text{Tr} Q^{-1} \sum_{i=0}^N F_i^T(x_{i+1}^*|\hat{X}) M \left( F_i^T(x_{i+1}^*|\hat{X}) \right)^T + \frac{1}{2} \sum_{i=0}^N \left[ \log |Q^{-1}| + \log \left| F_i^T(x_{i+1}^*|\hat{X}) \right| + \log \left| F_i^T(x_{i+1}^*|\hat{X}) \right| \right]
\]

where \( \tilde{L} \) is a function independent of \( Q \),

\[
M = \left( -J I \right) F_i^T(x_{i+1}^*|\hat{X}) + \left( \bar{x}_{i+1}^* - F_i(x_{i+1}^*|\hat{X}) \right) \left( \bar{x}_{i+1}^* - F_i(x_{i+1}^*|\hat{X}) \right)^T
\]

and \( J_i \) is the Jacobian of \( F_i(x, u) \) evaluated at \( x = x_{i+1}^* \). The variables \( \bar{x}_{i+1}^* \), \( \bar{x}_{i+1} \) and \( \bar{P}_{i+1}^* \) are obtained if the augmented state vector \( \xi_i = (x_{i+1}^T, x_i^T)^T \) with the new model \( \xi_{k+1} = (x_{k+1}, u_k) \) is used in the EKS. That is, the EKS calculates

\[
\xi_{i+1}^* = \left( x_{i+1}^T, x_i^T \right)^T \text{ and } \bar{P}_{i+1}^* = \begin{pmatrix} P_{i+1,1}^* & P_{i+1,2}^* \\ P_{i+1,2} & P_{i+1,1} \end{pmatrix} \]

where \( x_{i+1}^* \), \( x_i^* \), \( P_{i+1,1}^* \), \( P_{i+1,2}^* \) and \( P_{i+1,1} \) are the first and second order moments of the smoothed \( \bar{x}_{i+1} \) and \( x_i \) respectively.

**The M-step**

Take the derivative of \( \Gamma(Q; Q_k) \) with respect to \( Q^{-1} \) and let the result be equal to zero to get the solution in the maximisation step according to

\[ Q_{k+1} = \frac{1}{N-1} \sum_{i=0}^N F_i^T(x_{i+1}^*|\hat{X}) M \left( F_i^T(x_{i+1}^*|\hat{X}) \right)^T \]

**Two Alternative Algorithms**

Two alternative methods are compared to the EM algorithm.

**Alg. 2: Minimisation of the path error**

1. Select diagonal \( Q_0 = \mathbb{R}^{n \times n} \)
2. Minimise \( \sum_{i=0}^N |s_i|^2 \) subject to \( \lambda_j > 0 \) and \( j = 1, \ldots, 4 \)
3. Calculate the noise \( v_i \) from (1a).
4. Let \( Q_{k+1} \) be the covariance matrix for \( v_k \).
5. If converged, stop; if not, set \( l = l + 1 \) and go to step 2.

**Alg. 3: Iterative covariance estimation with EKS**

1. Select \( Q_0 \in \mathbb{R}^{n \times n} \) and set \( l = 0 \).
2. Use the EKS with \( Q_0 \).
3. Calculate the noise \( v_k \) from (1a).
4. Let \( Q_{k+1} \) be the covariance matrix for \( v_k \).
5. If converged, stop; if not, set \( l = l + 1 \) and go to step 2.

**Application to Industrial Robots**

Consider the non-linear joint flexible two axes robot model

\[
x_1 = \begin{pmatrix} x_1 \\ x_4 \\ x_2 \\ x_3 \end{pmatrix}, \quad x_4 = \begin{pmatrix} M_{x}^{-1}(x_1) \left[ -C(x_1, x_2) - G(x_1) - A(x) + u \right] \\ M_{x}^{-1}(x_1) \left[ -C(x_1, x_2) - G(x_1) - A(x) + u \right] + \tau(x_1, x_2) \end{pmatrix}
\]

where \( x_1 = (x_1^T, x_2^T, x_3^T, x_4^T)^T \) and \( A(x) = D(x_4) - x_4 \). The model structure (1a) and (2) is obtained when an Euler forward approximation is used to discretise the robot model.

I. The model was simulated to get the control signal \( u \) and the measurements, i.e., the motor angles \( q_{d0} \) and the acceleration of the tool.

II. The true tool position, used in Alg. 2, was also calculated.

III. The three algorithms where applied to the data to get \( Q \).

IV. The three \( Q \)-matrices were used in an extended Kalman filter to obtain an estimate of the tool position.

V. The path errors \( e_i = \sqrt{\sum_{k} (x_{d0} - x_{i})^2 + (y_{d0} - y_{i})^2} \) for these estimates were used to compare the three algorithms.

**Result**

- Alg. 2 gives different solutions for different initial values.
- EM and Alg. 3 give consistent solutions for different initial values.
- The path error for EM is much lower than Alg. 2 and Alg. 3.
- EM converges in around 50 iterations.